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ON INSTABILITY OF EQUILIBRIUM WHEN THE FORCE FUNCTION IS NOT A MAXIMUM *

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Some theorems on the instability of equilibrium of a mechanical system when the force function is not a maximum are proved by using Poincaré's recurrence theorem and the principle of least action in Jacobi's form. The behavior of the trajectories of the system as a whole is examined.

1. We consider a real autonomous system of equations

$$\mathbf{x}^{\cdot} = \mathbf{f}(\mathbf{x}) \tag{1.1}$$

Here $\mathbf{x}$ is a point in the n-dimensional phase space R^n, with coordinates $(x_1, \ldots, x_n)$, $\mathbf{f}$ is a vector-valued function on R^n, defined by the collection $(f_1, \ldots, f_n)$ of real functions on R^n. We assume that the solutions of system (1.1) are defined for $t \geqslant 0$ for any initial data. All functions encountered below are assumed continuous and to have continuous first-order partial derivatives. If system (1.1) satisfies the incompressibility condition

$$\sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i} \equiv 0$$

and V is a domain invariant relative to the flow generated by system (1.1), then the following statement is valid.

Theorem 1. If a function $W(\mathbf{x})$ exists such that the inequality

$$W^{\cdot}(x) = \sum_{i=1}^{n} \frac{\partial W}{\partial x_i} f_i(\mathbf{x}) \leqslant 0 \tag{1.2}$$

is fulfilled in domain V and $W^{\cdot}(\mathbf{x}) \not\equiv 0$ in V, then V is an unbounded set in R^n.

Proof. Let $\mathbf{x}(t, \mathbf{x}_0)$ be a solution of system (1.1), starting at $\mathbf{x}_0 \in R^n$, so that $\mathbf{x}(0, \mathbf{x}_0) = \mathbf{x}_0$. Let us assume that domain V is bounded and, consequently, has a finite measure. Then all the hypotheses of Poincaré's recurrence theorem /1/ are fulfilled for V. This signifies that for almost every point $\mathbf{x}_0 \in V$ there exists a sequence of times $\{\tau_n\}$ such that the equalities

$$\lim_{n\to\infty} \tau_n = \infty, \quad \lim_{n\to\infty} \mathbf{x}(\tau_n, \mathbf{x}_0) = \mathbf{x}_0, \quad \tau_1 = 0 \tag{1.3}$$

are valid. We denote the set of these points by V_1. Thus, the measure of V_1 equals the measure of V. Since $W^{\cdot}(\mathbf{x}) \not\equiv 0$, in domain V we can find a point $\mathbf{x}_0$ and a neighborhood B of it such that the inequality $W^{\cdot}(\mathbf{x}_0) < 0$ is valid in B, and we can take it that $\mathbf{x}_0 \in V_1$.

We consider the equality

$$W(\mathbf{x}(\tau_n, x_0)) - W(\mathbf{x}_0) = \int_0^{\tau_n} W^{\cdot}(\mathbf{x}(t, \mathbf{x}_0))\, dt \tag{1.4}$$

Here $\{\tau_n\}$ is chosen in accord with (1.3). Using (1.3) and the continuity of W, we obtain

$$\lim_{n\to\infty} (W(\mathbf{x}(\tau_n, \mathbf{x}_0)) - W(\mathbf{x}_0)) = 0$$

Then from (1.4) it follows that

$$\lim_{n\to\infty} \int_0^{\tau_n} W^{\cdot}(\mathbf{x}(t, \mathbf{x}_0))\, dt = 0 \tag{1.5}$$

Equality (1.5) leads to a contradiction. Indeed, from (1.2) it follows that the sequence in the right hand side of (1.4) is monotonic and, consequently, has a limit. On the other hand, since $\mathbf{x}_0 \in B$, we have $W^{\cdot}(\mathrm{x}(0, \mathrm{x}_0)) = W^{\cdot}(\mathbf{x}_0) < 0$ and, consequently,

$$\lim_{n\to\infty} \int_0^{\tau_n} W^{\cdot}(\mathbf{x}(t, \mathbf{x}_0))\, dt < 0$$

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From the contradiction obtained it follows that V is an unbounded set.

Corollary. Let the equilibrium position of system (1.1) occur in the closure of V. Then, if in any neighborhood of the equilibrium position a point x_0 exists such that $\mathbf{x}_0 \in V$ and $W^{\cdot}(\mathbf{x}_0) < 0$, then this equilibrium position is unstable. Furthermore, for any $M > 0$ and any $\delta > 0$ a system trajectory exists starting in the δ-neighborhood of the equilibrium position and leaving the M-neighborhood of the equilibrium position after a finite time.

Proof. We assume the contrary. Then $\delta > 0$ exists such that the inequality $\|\mathbf{x}(t, \mathbf{x}_0)\| < M$ is valid when $\|x_0\| < \delta$. Having considered the union of the trajectories starting in the δ-neighborhood of the equilibrium, we obtain a bounded domain B invariant relative to the flow generated by the trajectories of system (1.1). Then the hypotheses of Theorem 1 are fulfilled for $B \cap V$. Thus, $B \cap V$ must be an unbounded set, which is impossible because B is bounded.

From the Corollary if follows that a Liapunov function not depending explicitly on time and not being a first integral of the system cannot be constructed for systems satisfying the incompressibility condition if the equilibrium is stable. This fact was noted for certain control systems not satisfying the incompressibility condition, although it does not obtain in the general case /2/.

Chetaev's instability theorem /3/ as applied to systems satisfying the incompressibility condition can be modified under the assumption that $W^{\cdot} \geqslant 0$ in the domain $C(W > 0)$, and $W^{\cdot} \not\equiv 0$ in domain $B \cap C$, where B is any neighborhood of the equilibrium. Chetaev's instability theorem with the application of two functions /4/ can be modified analogously.

2. Let us consider a holonomic conservative mechanical system with n degrees of freedom $(q_1, \ldots, q_n)$. Let T be the system's kinetic energy and U be the potential energy. We reckon that the equilibrium position, possibly unisolated, coincides with the origin O of a configuration space. We assume that the inequality

$$\partial U / \partial q_1 \geqslant 0 \qquad (2.1)$$

is valid in some neighborhood B of point O and that in any neighborhood $B_1 \subset B$ of point O we can find a point for which the strict inequality (2.1) is valid. We assume as well that $\partial T / \partial q_1 \geqslant 0$.

Theorem 2. The equilibrium position of the system described above is unstable.

Proof. We consider the system's equations of motion in the Hamiltonian form. Let $P(A)$ be the projection of set A from the phase (q, p)-space onto the configuration q-space. We assume the stability of the equilibrium position. Then a neighborhood V of point O, bounded and invariant relative to the flow generated by the system's phase trajectories, exists such that $P(V) \subset B$. We consider the function $W(q, p) = p_1$. Allowing for (2.1), we obtain

$$W^{\cdot} = -\partial H / \partial q_1 = -\partial T / \partial q_1 - \partial U / \partial q_1 \leqslant 0$$

and $W^{\cdot} \not\equiv 0$ in domain V. Then Theorem 1 tells us that V is an unbounded set, which contradicts the choice of V. Thus, the system's equilibrium position cannot be stable. Theorem 2 has been proved.

3. We now assume that the coefficients $a_{ij}(\mathbf{q})$ of forms T and U are functions of class C^1. Let $U \leqslant 0$ everywhere on R^n. We denote the equilibrium position by O. The following theorem is valid under these assumptions.

Theorem 3. For any $M > 0$ and $\delta > 0$, there exist $\|\mathbf{q}_0\| \leqslant \delta$ and $\|\mathbf{q}_0^{\cdot}\| \leqslant \delta$ such that a trajectory of the mechanical system with these initial data, perhaps more than one, exists for some t, $\|\mathbf{q}(t)\| = M$.

Following an idea of Hagedorn /5/, we obtain the family of trajectories needed from the priniciple of least action /5/. In order to get rid of the assumption on strictness of the maximum of U, essential in /5/, we change not the equilibrium position, as in /5/, but the variational problem itself. This leads us as well to results of a nonlocal nature. In the proof of Theorem 3 we shall use facts concerning an elliptic positive variational problem (see Sections 29, 51, 54 in Vol.1 of /6/) /7/.

Proof. At first we assume that $U, a_{ij} \in C^\infty$. We fix $h > 0$ and we consider a variational problem with fixed endpoints

$$I(C) = \int_0^Q 2(h - U)^{1/2} \Big(\frac{1}{2} \sum_{ij=1}^n a_{ij} q_i' q_j'\Big)^{1/2} ds = \min \qquad (3.1)$$

Here s is the arc length on curve $C(O, Q)$. We solve the problem in the class of curves admitting of parametrization that is piecewise-smooth in s. By the principle of least action the solutions of problem (3.1) determine the trajectories of the corresponding mechanical system, along which $T + U = h$. Consider balls B, B_1, B_2, B_3 centered at O with radii $r < r_1 < r_2 < r_3$.

We fix a point $Q \in \partial B_{3}$ (∂B_3 is the boundary of set A). We define a function $e(x)$ thus

$$e(x) = \begin{cases} 0, & x \leqslant r^2 \\ \exp[-(x-r^2)^{-2}], & x > r^2 \end{cases}$$

As is well known, $e(x) \in C^\infty$. Having replaced U in problem (3.1) by the function

$$U_\lambda = U + \lambda e(\Sigma q_i^2),\ \lambda < 0,\ U_\lambda \in C^\infty$$

we once again obtain an elliptic positive problem

$$I_\lambda(C) = \min \tag{3.2}$$

Since (3.2) is a positive problem, the minimizing /5/ sequence of curves C_ν ($\nu = 1, 2, \ldots$) joining in B_3 the points O and Q automatically exist. By α we denote the lower bound for T when $q \in B_3$ $\|q^\cdot\| = 1$, and by γ the lower bound for $e(\|q\|^2)$ when $q \in \overline{B_2 / B_1}$. We consider an admissible curve $C(O, Q)$ in B_3 intersecting the boundaries ∂B_1 and ∂B_2. Let s_2 be the lower bound of the values of parameter s for which curve C intersects ∂B_2 and s_1 be the upper bound of the values of parameter s for which C intersects ∂B_1: $Q(s_1) \in \partial B_1$, $Q(s_2) \in \partial B_2$. Then the value of functional I_λ on the part of curve C from s_1 to s_2 can be bounded from below by the quantity

$$\varkappa = 2(r_2 - r_1)(|\lambda| \alpha\gamma)^{1/2}$$

We select λ so as to fulfil the inequality $I_\lambda(\overline{OQ}) < \varkappa$ ($\overline{OQ}$ is a segment).

This can be done. Indeed, by the construction of I_λ, $I_\lambda(\overline{OQ})$ is in fact independent of λ, while by choice of λ we can make $\varkappa$ arbitrarily large. We fix the λ needed. Having replaced the curves from the sequence C_ν ($\nu = 1, 2, \ldots$) intersecting boundaries ∂B_1 and ∂B_2 by the segment $\overline{OQ}$, we obtain a new minimizing sequence located in B_2, i.e., strictly inside relative to B_3. By a well-known theorem of the calculus of variations (see Lemma 51.31 in Section 51 of Vol.1 of /6/) an extremal $C(O, Q)$ of length s_0 exists for problem (3.2), furnishing the minimum of functional I_λ.

The parameter time is introduced on curve $C(O, Q)$ by the formula

$$t = \int_0^Q (h - U)^{-1/2} \left(\frac{1}{2} \sum_{ij=1}^{n} a_{ij} q_i' q_j'\right)^{1/2} ds$$

obviously the point Q is reached in a finite time t_0. Let $t_1 \leqslant t_0$ be the time in which the boundary ∂B is reached. On the interval $0 \leqslant t \leqslant t_1$ the functions $q_1(t), \ldots, q_n(t)$ specifying curve C determine the solutions of the Lagrange equations of the original mechanical system. This follows from the principle of least action and from the construction of functional I_λ. By β we denote the smallest eigenvalue of matrix $\|a_{ij}(0)\|$. By virtue of the energy integral the inequality $\|q^\cdot(0)\| \leqslant (2h / \beta)^{1/2}$ is valid for the initial velocity $q^\cdot(0)$. Thus it is clear that by letting h tend to zero we obtain a family of trajectories establishing the instability of the equilibrium. We set $M = r$.

In order to carry the result over to the case when $U \in C^1$, $a_{ij} \in C^1$ we make use of a standard procedure based on a theorem of Arzelà /8/. As a matter of fact, we approximate the functions U and a_{ij} by functions U_k, $a_{ij}^k \in C^\infty$ $(k = 1, 2, \ldots)$ on compactum $\bar{B}_3$ in such a way that the convergence to U and a_{ij} as in the C^1-topology. We can take it that $U_k \leqslant 0$ on $\bar{B}_3$ and that the forms T_k are positive definite /8/. Repeating the preceding arguments for each k, we obtain a family of curves C_k $(k = 1, 2, \ldots)$. It can be shown that this family of curves satisfies the hypotheses of Arzelà's theorem. Thus, we can select a subsequence $\{C_k'\}$ from the sequence $\{C_k\}$, converging to a trajectory of the original mechanical system. The choice of this subsequence may not be unique and, consequently, there can be several trajectories with the given initial conditions. This is understandable since the condition $U \in C^1$, $a_{ij} \in C^1$ does not guarantee the uniqueness of the solutions of the differential equations of motion. The theorem has been proved.

Note 3.1. In the case of a local minimum B_3 must be chosen such that $U(q) \leqslant 0$ when $q \in \bar{B}_3$.

We consider a plane π in R^n, passing through O. By π_1 and π_2 we denote the half-spaces into which R^n is divided by plane π. Let $P: R^n \to R^n$ be a symmetry mapping relative to plane π and $DP: T(R^n) \to T(R^n)$ be a mapping tangent to P. We define functions U^* and a_{ij}^* $(i, j = 1, \ldots, n)$ as follows

$$U^* = \begin{cases} U(q), & q \in \bar{\pi}_1 \\ U(P(q)), & q \in \bar{\pi}_2 \end{cases}, \quad a_{ij}^* = \begin{cases} a_{ij}(q), & q \in \bar{\pi}_1 \\ a_{ij}(P(q)), & q \in \pi_2 \end{cases}$$

We assume that $U^*, a_{ij}^* \in C^2$. Regarding the form

$$T^* = \sum_{ij} a_{ij}^* q_i^{\cdot} q_j^{\cdot}$$

(T^* can be looked upon as a function on the space $T(R^n) = R^n \times R^n$ tangent to R^n) we assume that $T^* \circ DP = T^*$, where $T^* \circ DP$ is a composition of mappings. For example, let

$$T = \sum_{i=1}^{n} a_i q_i^{\cdot 2}, \quad a_i = \text{const}$$

Then for any location of plane π the condition $T^* \circ DP = T^*$ holds automatically for T^*. The following theorem is true under these assumptions.

Theorem 4. If $U \leqslant 0$ when $q \in \bar{\pi}_1$, then the system's equilibrium is unstable and for any $M > 0$ and $\delta > 0$ there exists a trajectory of the system with initial data $\|\mathbf{q}(0)\| < \delta$, $\|\mathbf{q}^{\cdot}(0)\| < \delta$, such that $\|\mathbf{q}(t)\| = M$ for some t.

Proof. In problem (3.1) we replace U by U^* and we consider the corresponding variational problem $I^*(C) = \min$, taking it that $Q \in \partial B \cap \pi$. By hypothesis, $U^* \leqslant 0$ on R^n and $U^* \in C^2$. All the arguments relevant to problem (3.1) are valid also for the problem being analyzed. Indeed, to obtain the extremal $C(O, Q)$ in problem (3.1) the condition $U \in C^\infty$, $a_{ij} \in C^\infty$ is unnecessary since $U \in C^2$ suffices for the application of the lemma on the minimizing sequence of curves /5/. Thus, by a verbatim repetition of the arguments relevant to (3.1), we find the minimizing extremal $C^*(O, Q) \subset B_2$ of length $\bar{s}$ of the altered variational problem $I_\lambda^*(C) = \min$; λ is chosen as in problem (3.2).

By s_0 we denote the value of parameter s for which curve C first intersects ∂B. We define a set $N \subset [0, \bar{s}]$ as follows. We take $s \in N$ if the point $C - Q(s) \in \pi$ and if in any neighborhood of s there exists s' such that $Q(s') \notin \pi$. At first we assume that $N \neq \varnothing$ and show that N consists of a finite number of elements. Indeed, otherwise a point s^1, the limit point for N would exist. By construction the set N is closed, so that $s' \in N$. We choose $\rho > 0$ in accordance with the local existence theorem in the calculus of variations (see Theorem 29.5 in Chapter 2 of Vol.1 of /6/). Then $s' \in \bar{N}$ exists such that $|s' - s^1| < \rho/2$. Let $s' < s^1$. Then, if the arc $C^*(Q(s')\ Q(s^1))$ does not lie in plane π, another extremal $C^{**} = P(C^*(Q(s'), Q(s^1)))$ exists joining $Q(s')$ and $Q(s^1) \in \pi$.

Indeed, from the theorem's hypotheses it follows that $I_\lambda^*(C^*(Q(s'), Q(s^1))) = I_\lambda^* \cdot (C^{**}) = \min$. Both these extremals (recall that s is the length) lie in a ball of radius ρ, which is impossible by virtue of the choice of ρ, so that necessarily $C^*(Q(s'), Q(s^1)) \subset \pi$ and $(s', s^1) \cap N = \varnothing$. But since s^1 is a limit point for N, there exists $s'' > s^1$ such that $s'' \in N$ and $s'' - s^1 < \rho/2$. As above we can prove that $C^*(Q(s^1), Q(s'')) \subset \pi$ and $(s^1, s'') \cap N = \varnothing$. Thus, $s^1 \notin N$, which contradicts the choice of s^1. Consequently, $N = \{s_1 \leqslant s_2 \leqslant \ldots \leqslant s_k\}$. The arc $C^*(Q(s_j), Q(s_{j+1}))$ is located either in π_1 or in π_2 ($Q(s) \notin \pi$ for $s \in (s_j, s_{j+1})$) or in π. Having mapped the part of curve $C^*(O,Q)$, lying in π_2, into π_1 symmetrically relative to plane π, we obtain a piecewise smooth curve $C^{**}(O, Q) \subset \bar{\pi}_1$ since smoothness can be violated only at the points of set N. But from the definitions of U^* and a_{ij}^* and from the fact that $T^* = T^* \circ DP$ it follows immediatley that $I_\lambda^*(C^*(O, Q)) = I_\lambda^*(C^{**}(O,Q)) = \min$, so that $C^{**}(O,Q)$ is necessarily a smooth curve, being, as is $C^*(O,Q)$, the minimizing extremal for the altered problem $I_\lambda^* = \min$ relative to domain B_3, while its part up to the first intersection with ∂B, namely, $C^{**}(O, Q(\bar{s}_0))$ defines an extremal, possibly not minimizing, for the problem $I^*(C) = \min$. Indeed, the problem's Euler equations coincide with the Euler equations of the problem $I_\lambda^* = \min$ on set $\bar{B}$. But the Euler equations for the variational problem corresponding to the original mechanical system coincides on $\overline{\pi_1 \cap B}$ with the Euler equations for the problem $I_\lambda^* = \min$ by the construction of U^* and a_{ij}^*, and since $C^{**}(O, Q(\bar{s}_0)) \subset \overline{B \cap \pi_1}$, $C^{**}(O, Q(\bar{s}_0))$ determines a trajectory of the original mechanical system.

However, if $N = \varnothing$, then the original system's trajectory is found directly by the principle of least action from the curve $C^*(O, Q(\bar{s}_0))$ if $C^*(O, Q) \subset \bar{\pi}_1$ or from the curve $P(C^*)$ if $C^*(O, Q(\bar{s}_0)) \subset \bar{\pi}_2$. Letting h tend to zero and repeating for each h the arguments presented, we obtain the desired family of trajectories. The theorem is proved.

Within the hypotheses of Theorem 4 we can find, for example, a system in which the a_{ij} are independent of q_1, $U = q_1^3 + F(q_2, \ldots, q_n)$, $F \leqslant 0$, and the plane $q_1 = 0$ is chosen as π (functions a_{ij} can be taken symmetric relative to plane $q_1 = 0$). Let us return to the case when U has a strict maximum at the equilibrium, $U \in C^2$, and $a_{ij} \in C^2$ $(i, j = 1, \ldots, n)$. The instability of the equilibrium was established in /5/ by using the family of trajectories along which the energy integral's constant h is positive. Let us prove the following theorem.

Theorem 5. For any $M > 0$ and $\delta > 0$ there exists, if the maximum is global, a trajectory of the system with initial data $\|\mathbf{q}(0)\| < \delta$, $\|\mathbf{q}^{\cdot}(0)\| < \delta$, such that along it $h < 0$ and $\|\mathbf{q}(t)\| = M$ for some t.

Proof. We consider variational problem (3.1) for $h < 0$. This problem is elliptic and positive in domain $M_h = \{Q \mid U(Q) < h\}$. By U_ρ we denote the lower bound of U on compactum

B_3 / B_ρ, where B_ρ is a ball of radius ρ centered at O. We fix some $\delta > 0$ such that $B_\delta \subset B$ and we consider the altered variational problem $I_\lambda = \min$, taking the endpoints $Q_1 \in \partial B_\delta$ and $Q_2 \in \partial B$ as being fixed. We set $U_\lambda = U + \lambda [e(x) + e_1(x)]$, where $\lambda < 0$ and function e_1 is defined by the formula

$$e_1(x) = \begin{cases} 0, & x \geqslant \delta^2 \\ \exp[-(x-\delta^2)^{-2}], & x < \delta^2, \end{cases} \quad x = \sum_{i=1}^{n} q_i^2$$

We choose h from the inequality $U_\rho < h < 0$, where $\rho = \delta / 2$. It is clear that $\overline{B_3 / B_\rho} \subset M_h$. Arguing as for problem (3.1) (instead of B_3 and B we examine $M_h \cap B_3$ and B / B_δ, respectively), we conclude the existence of an extremal $C_\delta(Q_1, Q_2)$ for (3.2) relative to domain $M_h \cap B_3$ for λ sufficiently large in modulus. Let s_2 be the lower bound of the values of parameter s, for which $C_\delta(Q_1, Q_2)$ intersects ∂B, and let s_1 be the upper bound of the values of parameter $s < s_2$, for which $C_\delta(Q_1, Q_2)$ intersects ∂B_δ. Since by its construction $U_\lambda = U$ on the set $\overline{B / B_\delta}$, the Euler equations for problem (3.1) coincide on $\overline{B / B_\delta}$ with the Euler equations for problem (3.2) and, consequently, the arc $C_\delta(Q(s_1), Q(s_2))$ determines an extremal, possibly not minimizing, for problem (3.1). Further, by the principle of least action we find a trajectory of the original mechanical system ($Q(s_1)$ is the initial point). Letting δ tend to zero, we obtain the desired family of trajectories. Indeed, $h \to 0$ as $\delta \to 0$ and $q(0) \to 0$ since $q(0) = Q(s_1) \in \partial B_\delta$, so that $q^{\cdot}(0) \to 0$ by virtue of the energy integral. We select R as M. The theorem is proved.

A note analogous to Note 3.1 can be made regarding Theorems 4 and 5 when the maximum is local.

The results obtained can be applied with appropriate changes to the inversion of Routh's theorem /5/.

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